Announcements

1) HW 2 due Thursday

Example from last time: image compression

"large" file represented as a matrix

$$\left[\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{array}\right]$$

average over 2x2 blocks to get a smaller file

We get

$$\begin{bmatrix} \frac{14}{4} & \frac{22}{4} \\ \frac{23}{2} & \frac{54}{4} \end{bmatrix}$$

Want: express this process using

matrix multiplication

Matrix Multiplication

The rules: Given an $m \times n$ matrix A (m rows, n columns), we can multiply A on the

1) left by a $k \times m$ matrix (same number of columns as A has rows) to get a $k \times n$ matrix

2) right by a $n \times \ell$ matrix (same number of rows as A has columns) to get an $m \times \ell$ matrix

The idea : dot products

Exumple 1:

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 5 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 8 \\ -1 & 15 \\ -6 & 4 \end{bmatrix}$$

$$B \cdot A = \begin{bmatrix} 3 & 8 \\ -1 & 15 \\ -6 & 4 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3(-1)+8\cdot 2 & 3\cdot 3+8\cdot 5 \\ (-1)(-1)+15\cdot 2 & (-1)\cdot 3+15\cdot 5 \\ (-6)(-1)+4\cdot 2 & (-6)\cdot(3)+4\cdot 5 \end{bmatrix}$$

1st row of B
dot product with
1st column of A

1st row of B
dot product with
2nd column of A

$$\begin{bmatrix} \boxed{3(-1)+8\cdot 2} & \boxed{3\cdot 3+8\cdot 5} \\ (-1)(-1)+15\cdot 2 & (-1)\cdot 3+15\cdot 5 \\ (-6)(-1)+4\cdot 2 & (-6)\cdot(3)+4\cdot 5 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 49 \\ 31 & 42 \\ 14 & 2 \end{bmatrix}$$

a 3x2 matrix

In general:

If A is $m \times n$ and B is $k \times m$,

$(BA)_{i,j}$ = entry in the i^{th} row, j^{th} column of BA

$(1 \leq i \leq k, 1 \leq j \leq n)$

= dot product of the i^{th} row of B with the j^{th} column of A

= a number.

For example, $(BA)_{3,5}$ = dot product of the 3^{rd} row of B with the 5^{th} column of A

Back to compression:

we started with a 4×4 matrix A and ended with a 2×2 matrix D. Since we changed <u>both</u> the rows and columns of A (in terms of their number), we need to find matrices B and C with

$$BAC = D$$

$B = 2\times 4$
$C = 4\times 2$

Step 1: Find B, make it average - almost!

$$\overset{B}{\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} \end{bmatrix}} \left[\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} \frac{6}{4} & \frac{8}{4} & \frac{10}{4} & \frac{12}{4} \\ \hline \frac{22}{4} & \frac{24}{4} & \frac{26}{4} & \frac{28}{4} \end{array}\right]$$

Step 2: Find C, make it add the numbers in the blue 1×2 blocks

$$\left[\begin{array}{cc|cc} \frac{6}{4} & \frac{8}{4} & \frac{10}{4} & \frac{12}{4} \\ \hline \frac{22}{4} & \frac{24}{4} & \frac{26}{4} & \frac{28}{4} \end{array}\right] \overset{C}{\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}}$$

$$= \begin{bmatrix} \frac{14}{4} & \frac{22}{4} \\ \frac{46}{4} & \frac{54}{4} \end{bmatrix} \quad ✓$$

Example 2:

Start with a 3x3 matrix

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \hline 7 & 8 & 9 \end{array}\right] = A$$

average over the orange blocks to again get a 2x2 matrix

$$\begin{bmatrix} \frac{1+2+4+5}{4} & \frac{3+6}{2} \\ \frac{7+8}{2} & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & \frac{9}{2} \\ \frac{15}{2} & 9 \end{bmatrix} = D$$

Want B, C where

$$BAC = D$$

Step 1: Find B, make it "average"

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{2} & \frac{7}{2} & \frac{9}{2} \\ 7 & 8 & 9 \end{bmatrix}$$

Step 2: Find C, make it "correct" any errors from B

$$\begin{bmatrix} \frac{5}{2} & \frac{7}{2} & \frac{9}{2} \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{12}{4} & \frac{9}{2} \\ \frac{15}{2} & 9 \end{bmatrix} = \begin{bmatrix} 3 & \frac{9}{2} \\ \frac{15}{2} & 9 \end{bmatrix}$$

✓

Easy Path

Vectorization: take an $n \times m$ matrix, turn it into an

$nm \times 1$ vector.

Example 3

$$A = \begin{bmatrix} -6 & 55 \\ 4 & 3 \end{bmatrix}$$

2x2 matrix = 4 entries

vector should be 4x1

$$v = \begin{bmatrix} -6 \\ 55 \\ 4 \\ 3 \end{bmatrix}$$

remembers all the entries of A

The position of the numbers in A is ambiguous just given v.

How do we do this using matrices?

Want a 4x1 vector from example.

$$(4\times 2)\ \overset{2\times 2}{A}\ (2\times 1)$$

$$A\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}-6 & 55\\4 & 3\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$$

$$=\begin{bmatrix}-6\\4\end{bmatrix}$$

$$A\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}55\\3\end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -6 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} -6 \\ 4 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 55 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 55 \\ 3 \end{bmatrix}$$

add the outputs to get $\begin{bmatrix} -6 \\ 4 \\ 55 \\ 3 \end{bmatrix}$

This procedure vectorizes A, but we don't get the vector we wanted – the order is off.

To fix the order, multiply by a permutation matrix, in this case

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -6 \\ 4 \\ 55 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -6 \\ 55 \\ 4 \\ 3 \end{bmatrix} \quad ✓$$

If we wanted to average the entries of A, we could multiply our vector by

$$\overset{1 \times 4}{\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}} \begin{bmatrix} -6 \\ 55 \\ 4 \\ 3 \end{bmatrix}$$

$$= \frac{1}{4}(-6+55+4+3)$$

= the average!

We could write this process as (forgetting the permutation matrix)

$$\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} A \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

= average of the entries of A.

Back to example 2:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \hline 7 & 8 & 9 \end{array}\right] = A$$

Step 1: vectorize A.

$$\overset{9\times 3}{\begin{bmatrix} I_3 \\ O_3 \\ O_3 \end{bmatrix}} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \overset{3\times 1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

where $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (identity matrix)

and $O_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (zero matrix)

$$\begin{bmatrix} I_3 \\ 0_3 \\ 0_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} I_3 \\ 0_3 \\ 0_3 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 4 \\ 7 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0_3 \\ I_3 \\ 0_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0_3 \\ I_3 \\ 0_3 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 5 \\ 8 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0_3 \\ 0_3 \\ I_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0_3 \\ 0_3 \\ I_3 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 6 \\ 9 \end{bmatrix}$$

add all vectors!

We get $$\begin{bmatrix} 1 \\ 4 \\ 7 \\ 2 \\ 5 \\ 8 \\ 3 \\ 6 \\ 9 \end{bmatrix} = v$$

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \hline 7 & 8 & 9 \end{array}\right] = A$$

Averaging gets a 2x2 matrix.

We need to multiply v by

(2×9) $\underset{9\times 1}{v}$ (1×2)

4 x 9

$$\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 7 \\ 2 \\ 5 \\ 8 \\ 3 \\ 6 \\ 9 \end{bmatrix}$$

Cheat - make a 4x1 vector that contains the averaged entries of A. Each row should average a particular "block" of A.

We get

$$\begin{bmatrix} 3 \\ \frac{15}{2} \\ \frac{9}{2} \\ 9 \end{bmatrix}$$

recover the 2x2 matrix by reversing the vectorization process.

General Implementation

To average and n×m matrix into k×l "blocks", first vectorize:

$$\sum_{i=1}^{m} I_{n,i} A e_i^{(m)}$$

where $e_i^{(m)} = m \times 1$ vector with a one in the $i^{\underline{th}}$ row and zeroes in all other rows

$$\sum_{i=1}^{m} I_{n,i} A e_i^{(m)}$$

where $e_i^{(m)}$ = m×1 vector with a one in the $i^{\underline{th}}$ row and zeroes in all other rows

and

$I_{n,i}$ = nm×n matrix with I_n block in the $i^{\underline{th}}$ position, zeros every where else

Then average the $nm \times 1$ vector into a $kl \times 1$ vector by choosing a $kl \times nm$ matrix with, in each row, nonzero entries at the corresponding row positions to the numbers you want to average in the $nm \times 1$ vector, the entries all being the number of entries being averaged. Last step! undo vectorization.

Example 4:

$$A = \left[\begin{array}{c|cc|cc} 5 & -1 & -2 & 0 & 3 \\ 8 & 4 & 11 & 6 & -3 \end{array}\right]$$

make A into a 1x3 matrix by averaging in the indicated blocks.

Express this procedure using matrices.

Step 1: vectorize A

need a 10x1 vector

$(10\times 2)\ A\ (5\times 1)$

$$\begin{bmatrix} I_2 \\ 0_2 \\ 0_2 \\ 0_2 \\ 0_2 \end{bmatrix} A \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} I_2 \\ 0_2 \\ 0_2 \\ 0_2 \\ 0_2 \end{bmatrix} \begin{bmatrix} 5 & -1 & -2 & 0 & 3 \\ 8 & 4 & 11 & 6 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 8 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Continue:

$$\begin{bmatrix} 0_2 \\ I_2 \\ 0_2 \\ 0_2 \\ 0_2 \end{bmatrix} A \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0_2 \\ 0_2 \\ I_2 \\ 0_2 \\ 0_2 \end{bmatrix} A \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -2 \\ 1\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0_2 \\ 0_2 \\ 0_2 \\ I_2 \\ 0_2 \end{bmatrix} A \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 6 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0_2 \\ 0_2 \\ 0_2 \\ 0_2 \\ I_2 \end{bmatrix} A \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ -3 \end{bmatrix}$$

Add all vectors to get

$$\begin{bmatrix}\left.\begin{matrix}5\\8\end{matrix}\right\}\text{average}\\ \text{average}\left\{\begin{matrix}-1\\4\\-2\\11\end{matrix}\right.\\ \left.\begin{matrix}0\\6\\3\\-3\end{matrix}\right\}\text{average}\end{bmatrix} = \text{vectorized } A$$

Step 2: Average to get a 3x1 vector. Express using matrices.

3×1

(3×10)

$$\begin{bmatrix}5\\8\\-1\\4\\-2\\11\\0\\6\\3\\-3\end{bmatrix}$$

$$\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}\end{bmatrix}\begin{bmatrix}5\\8\\-1\\4\\-2\\11\\0\\6\\3\\-3\end{bmatrix}$$

$$= \begin{bmatrix}\frac{13}{2}\\ 3\\ \frac{3}{2}\end{bmatrix}$$

Step 3 Undo vectorization - if you want!